Solving Mathematical Crosswords

by Derek W. Arthur (© 2001)

1. Introduction

Mathematical puzzles differ from word puzzles in several ways and the most obvious difference – the use of numbers rather than words – is perhaps not the most significant. There is no obvious counterpart to the 'plain' cryptic word puzzle, such as those that appear in daily newspapers. In the word domain, those who seek to move on to 'thematic' or 'advanced' puzzles can rely on an existing facility to decipher and solve clues. In the number domain, there is no corresponding preparation available and the novice solver is faced with a perplexing variety of puzzles from the outset.

Another significant difference is encountered from the start. In word puzzles it is normally feasible to read through the clues, to solve any one of them and then to develop the solution to others from that point, making use of the intersections of grid entries. In number puzzles it is normal for there to be only one clue (or perhaps a linked pair) that can be solved immediately; the solver must identify it to have any hope of making progress.

These puzzles resemble a game of chess where the 'opening' has been stipulated. The player has little choice in that opening phase, being forced to follow the path laid out by the setter. In the 'middlegame', there may be several developments possible and the solver must evaluate which are likely to be most successful. Then comes the 'endgame', which usually involves only straightforward calculations.

The aim of these notes is to set out the basic conventions (in Section 2), to elaborate useful opening strategies (in Section 3), to set out possible developments (in Section 4) and to leave the solver to master the endgame. The author will claim success if the reader has contrived to avoid resignation by this point. The final sections describe devices that may be employed by more challenging 'opponents'.

Some crossword enthusiasts object to the term cross*word* for these puzzles, arguing that they involve numbers rather than words. This is, of course, justifiable semantically, but does not diminish their status as logical challenges, not that a well constructed and clued word puzzle is any less precise in a logical sense. What is more important is the *cross*word nature and here it is arguable that the mathematical puzzle fulfils that objective better than the word puzzle. The intersection of across and down entries is very important, but there are often essential links between clues whose entries are distant in the grid, something that rarely occurs in word puzzles.

A more legitimate complaint is against the word 'mathematical'. Most modern puzzles are arithmetic in nature and do not require algebra or other mathematics of a similar level. Some of the early *Listener* puzzles demand much more, but puzzles of that type rarely occur now. One reason is that some of them are open to attack by computer and thereby become glorified jigsaw puzzles. For most puzzles the content of Sections 2–4 will suffice.

Just as there is no commonly agreed 'plain' mathematical puzzle, it is difficult to define what is meant by the phrase, especially in the light of the great variety in style and content. Most have a set of clues expressed in terms of symbols – usually roman letters – whose numerical values must be deduced in order to construct the grid entries. The clues normally use algebraic notation, explained in Section 2. These notes do not give any special treatment of hybrid puzzles where some clues are numerical and some word-based. These often deliver several development possibilities and hence are easier to solve, at least for someone well-versed in cryptic clues. Similarly, puzzles that do not even appear to be numerical until the solution is well advanced are given no special treatment, nor are word puzzles where grid entry depends on simple arithmetical calculations using the numerical value of letters (A = `1`, B = `2`, ...).

It is assumed that the reader has access to a calculator, preferably a scientific one that can display reasonably large numbers and offers square roots, powers and bracketing of sub-calculations. This does not imply that manual calculation should not to be used: the possession of a dictionary does not mean that every word in a puzzle must be referenced therein.

Crosswords are relatively free from jargon, although there are two terms that are worth defining at this point. The number to which a clue leads is the *answer* and will be assumed to be decimal, i.e., given in base 10. The number to be written into the grid is the *(grid) entry*. This may be different if the puzzle involves other number bases or some other transformation is required before entry.

The following typographical conventions are used in what follows. Mathematical text uses standard notation, with italic letters for symbols and a normal font for numbers, e.g., $n = 2$. Clues are given in (small) roman capitals, as in published puzzles. Numbers that are to be entered in the grid or a working record or read from a calculator are given in a fixed width font, as in the tables in the *Appendix*, e.g., WORD = `12345` or `12□□5` if the third and fourth digits are not yet known.

Finally, these notes include some examples from published puzzles. The symbols have usually been changed to disguise the source, in case the reader encounters the puzzle and wishes to tackle it afresh.

2. Basic Conventions

This section sets out some of the common notations and assumptions made by setters. These may be made explicit in the preamble to the puzzle, but their absence is usually not significant. The opportunity will be taken to add some definitions of the more mathematical concepts that frequently appear and to offer advice on some calculator techniques.

It is dangerous to make definitive statements since each of these puzzles tends to be different in some way from others and setters are forever investigating new ideas. Nonetheless, the author cannot recall a puzzle in which any of the numbers in use was negative. Zero has occurred, although not as a clue answer. Fractions have certainly put in an appearance, but rarely.

One common convention – critically important for the solver – is that no *answer* should start with a zero. Also, it is almost always the case that grid *entries* are all different although, in a puzzle with a variety of number bases, answers may be the same.

Algebraic notation

Clues should be assumed to conform to normal algebraic notation, unless stated otherwise. In particular, juxtaposition means multiplication. Thus, if $\mathrm{A} = 2$ and $\mathrm{B} = 3$, then $\mathrm{AB} = 6$, not 23. This saves space and makes the clues easier to read.

Division is quite rare, perhaps because it gives away quite a lot of information about common factors (see later). When it does occur, it is most commonly represented by A/B, rather than $\frac{\mathrm{A}}{\mathrm{B}}$ or $\mathrm{A} \div \mathrm{B}$.

Powers and roots

As we see in the next two sections, *powers* of numbers are extremely useful: they often provide the starting point for a solution. There are two notations, one mathematical and one used in computer programming. The choice usually depends on the capabilities of the puzzle's printer. The mathematical definition is

$$x^n = x \times x \times x \times \cdots \times x \qquad (n \text{ copies of } x).$$

This will be printed in one of the forms A^{B} or A^B or $\mathrm{AAA} \cdots \mathrm{AA}$ (B copies of A).

The opposite calculation, that of *roots*, occasionally appears in clues, but is required much more in the solution process. We spend as much time moving from the grid to clues as from clues to the grid, e.g., we may ask what number, raised to a known power, could produce some partially completed grid entry. What is required is the n^{th} root, written $\sqrt[n]{x}$:

$$\sqrt[n]{x} \times \sqrt[n]{x} \times \cdots \times \sqrt[n]{x} = x, \qquad (n \text{ copies of } \sqrt[n]{x}).$$

Before discussing how to evaluate these powers and roots, there are some mathematical conventions that may occasionally be used, one of which is useful if one's calculator has limited functionality:

$$x^{-n} = \frac{1}{x^n}, \qquad x^{1/n} = \sqrt[n]{x}, \qquad x^{m/n} = \sqrt[n]{x^m}.$$

The following examples illustrate these conventions:

$$2^{-3} = 1/2^3 = 1/8 = 0.125, \qquad 64^{1/3} = \sqrt[3]{64} = 4, \qquad 4^{3/2} = \sqrt{4^3} = \sqrt{64} = 8.$$

(Note that 'square roots' are usually written $\sqrt{x}$ rather than $\sqrt[2]{x}$.)

Some powers are given in one of the tables in the *Appendix*. Scientific calculators have keys to help calculate them, usually labelled $\boxed{x^y}$. Thus, pressing 5 $\boxed{x^y}$ 4 gives 625 for 5^4.

Roots are more problematic, since there are at least two different calculator conventions, while some calculators have no key for direct root-finding. It is for such calculators that the fractional power convention is particularly helpful. Each of the following should give the answer 3 for $\sqrt[5]{243}$ (on an appropriate calculator):

$$5\ \boxed{\sqrt[x]{y}}\ 243, \qquad 243\ \boxed{x^{1/y}}\ 5, \qquad 243\ \boxed{x^y}\ 0.2.$$

(The direct root facility often shares the same key as the power facility and is accessed by first pressing a special key, perhaps labelled '`shift`', '`inv`' or '`2nd`'.)

Factorials

A further shorthand connected with multiplication is the *factorial* notation: $n!$. This stands for the product of all whole numbers between 1 and n inclusive, e.g.,

$$4! = 1 \times 2 \times 3 \times 4 = 24, \qquad 10! = 1 \times 2 \times \cdots \times 9 \times 10 = 3628800.$$

Some calculators have a $\boxed{x!}$ key and a few factorials are given in one of the tables in the *Appendix*. It is clear from that table, if not from the examples above, that these numbers grow in size very rapidly, so that the appearance of a factorial in a puzzle usually provides valuable information.

Rules of precedence

When a clue contains a variety of operations: $+$, $-$, $\times$, $/$, ^, we require rules to determine the order of calculation. The following rules agree with up-to-date calculators, although some earlier versions do not obey them:

powering is done first;

multiplication and division are done next;

addition and subtraction are done last;

for two operations with the same precedence, such as addition and subtraction, work from left to right.

Thus,

$$\begin{aligned} 3 \times 5 - 8 \times 2^2 + 6 \times 7 &= 3 \times 5 - 8 \times 4 + 6 \times 7 \\ &= 15 - 32 + 42 \\ &= -17 + 42 \\ &= 25. \end{aligned}$$

These conventions can be overruled by the use of *brackets*. When these are inserted, their contents must be evaluated before anything outside. Thus

$$(3 \times 5 + 2) \times 4 = 17 \times 4 = 68, \quad \text{while} \quad 3 \times 5 + 2 \times 4 = 15 + 8 = 23.$$

Prime numbers

Puzzles often involve these directly, e.g., coding some by symbols and requiring the solver to deduce them. A *prime (number)* is a positive whole number whose only *divisors* (or *factors*) are 1 and the number itself, e.g, 7, 11, 101. By convention, 1 is not regarded as a prime, so that the smallest prime is 2. A list of all primes less than 10000 is given in the *Appendix*.

Since much of the solution process can involve an analysis of products, the factors of a number are extremely important. These can be deduced from its *prime factors*, which dictates a rôle for a piece of mathematics often called the *Fundamental Theorem of Arithmetic*. This states that every number larger than 2 can be written, in a unique way, as the product of prime numbers. When this is done in practice, repeated prime factors are usually collected together to give an appropriate power. Thus

$$756 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 7 = 2^2 \cdot 3^3 \cdot 7.$$

Prime factors for numbers up to 1000, expressed in this way, are given in a table in the *Appendix*.

For a larger number n, we can proceed as follows. Work through the prime numbers in order, starting with 2. Using each, repeatedly divide n for as long as the answer is a whole number, replacing n by that whole number. Store the number of successful divisions for that prime: it will be its power in the answer. Stop when 1 is reached. There are some shortcuts available. For example, there is no need to divide by 2 if the number is odd or by 5 if it doesn't end in 5. Also, there is no need to use a prime larger than $\sqrt{n}$: any number left by that time must be prime. The process can be speeded by storing the current number in the calculator's memory so that it can be retrieved after each failed division and by checking the current number against the table of primes in the *Appendix* once it is less than 10000. For example, for 21496293 we should expect to use primes no larger than 4636. We find (omitting unsuccessful divisions):

21496293/3=7165431, 7165431/3=2388477, 2388477/3=796159,
796159/7=113737,
113737/13=8749, 8749/13=673.

The table shows 673 to be prime, so $21496293 = 3^3 \cdot 7 \cdot 13^2 \cdot 673$.

The applications of this information in solving puzzles are wide-ranging. For example, if we know that Y = 14, the clue BAB/Y implies that at least one of A and B is divisible by 7 and at least one is divisible by 2. This information may be usable in some other clue or partial grid entry to fix A or B.

The prime factors are also useful in dealing with powers and roots. If a number is a perfect square, its prime factors must each occur an even number of times. Thus,

$$324 = 2^2 \cdot 3^4, \quad \text{and so} \quad \sqrt{324} = 2^1 \cdot 3^2 = 18.$$

This fact, and its extension to cubes and other powers, can often be used directly to glean valuable information.

An understanding of this factorisation, however, can be useful in other ways. One published puzzle involves triples of numbers, (α, β, γ), where $\sqrt{\alpha} + \sqrt{\beta} = \sqrt{\gamma}$. An assumption that this requires each number to be a perfect square will not lead to a solution. Consideration of prime factorisation suggests that each number could be a perfect square multiplied by an *identical* set of odd powers of primes. For example, $\alpha = 60$, $\beta = 135$, $\gamma = 375$ would be possible, since

$$\begin{aligned}\sqrt{60} + \sqrt{135} &= \sqrt{(3 \cdot 5) \cdot 2^2} + \sqrt{(3 \cdot 5) \cdot 3^2} \\ &= \sqrt{2^2}\sqrt{15} + \sqrt{3^2}\sqrt{15} = 2\sqrt{15} + 3\sqrt{15} \\ &= 5\sqrt{15} = \sqrt{15 \times 5^2} = \sqrt{375}.\end{aligned}$$

Algebraic manipulation

It is rare that puzzles require an explicit knowledge or use of algebraic manipulation, although this is not unknown. Section 6 gives some examples. It is, however, difficult to draw a line here. Some may consider that the square root calculation in the previous item requires formal algebra, while others may label it as common sense. (It uses the algebraic rule: $\sqrt{mn} = \sqrt{m} \times \sqrt{n}$.)

A similar example concerns $\mathrm{A} + \mathrm{B}$, where both are known to be divisible by n, say. Then the sum is also divisible by n, which many would regard as obvious. A formal justification states that we must have $\mathrm{A} = pn$ and $\mathrm{B} = qn$, for some p and q, so that $\mathrm{A} + \mathrm{B} = pn + qn = (p + q)n$ and hence has a factor n.

This fact can be of help. Suppose we know that $\mathrm{Z} = 13$. Then the answer for PUZ + ZLE must also be divisible by 13. If we have some partial entry such as `17□56`, we can deduce that the only multiple of 13 is `17056`.

3. Getting Started

There are two aspects to the initial stages of the solution: preparation, followed by a search for a clue or set of clues that can provide a starting point.

The Preamble

Since there is no commonly agreed standard puzzle of this type, each should provide some explanation of its construction and objectives. This will appear before the clues and is usually called the *preamble*. The solver should read this very carefully, taking note of every point in it, no matter how minor or redundant it may appear. A well-edited puzzle is likely to have no redundant information!

The preamble may state that no answer starts with zero. This is virtually compulsory and the absence of any such statement should not be interpreted as suggesting the opposite. Similarly, puzzles usually do not contain the same entry more than once. The absence of a statement to this effect is unlikely to mean that there are repeated entries, but this ‘rule’ is not as inflexible as the earlier one.

Take note of any comment about algebraic notation. There are puzzles where AB stands for the 2-digit number composed of A and B, but they indicate that fact or provide an alternative notation if multiplication is also used. Also, take note of whether the clues are labelled by the grid numbers or by symbols whose values are to be determined. This latter style allows some useful deductions and is discussed later.

Finally, look out for any unusual wording: it may be disguising some part of the puzzle. For example, a comment about "missing the point" may mean that fractions in decimal form are involved, with the point ignored.

Record-keeping

If, as usual, the puzzle contains symbols whose values are to be established, it is sensible to draw up a list of these symbols so that their values can be recorded – ideally in pencil – as they are found. If there is a well-defined set of numbers for these values, a list of those should be drawn up so that, at any time, the solver knows not only the current values but which values are available for the other symbols.

When drawing up these lists, it is worth noting whether any of the letters o, O, I, l, x or X occur, since they can be misread as 0, 1 or ×. Indeed, it is not unknown for even the most prestigious puzzles to contain typographical errors of this nature. Setters are discouraged from using these letters, but there may be good reason to ignore that advice, e.g., a setter trying to construct clues that spell words will find it difficult in the absence of the two vowels I and O. Should one arrive at a contradiction that seems inescapable, it may be possible to proceed by examining the clues involved to see if the problem can be resolved by adjusting an 'I' or an 'l' or some other likely culprit. The puzzle may turn out to be solvable, but more challenging than intended!

If the answers to the clues are to be processed before entry, e.g., expressed in a different number base, then it may be worthwhile drawing up a list of clue numbers to record these intermediate answers.

Starting the Solution

It was mentioned at the outset that the setter is likely to have created just one element of the puzzle from which the first deductions can be made. Solvers may have to search through the clues several times before finding this. The following seven devices are commonly used; sometimes more than one is required.

1. Small Numbers

If one of the clues consists of a pure power (perhaps written as an explicit product) for which the grid entry is only of length 2 there may be few possibilities. Thus, if the clue is P^4 or PPPP, we know that P is 2 or 3, since $2^4 = 16$ and $3^4 = 81$, while 4^4 exceeds 99.

The following examples are variants of extracts from published puzzles. The clue DOGGED, where each letter represents a different number in the range 1–26, has entry length 2. It is clear that all these letters must be small; trial and error shows that one of D or G must be 1 and the other is either 2 or 3. Consideration of other clues involving D and G allows progress to be made. In another case, $\mathrm{RU}^4\mathrm{M}^2$ has length 4. It is easy to see that U and M must be small. Further progress can be made since the entry intersects with that for another clue involving U and M.

2. Large Numbers

Powers can also be informative if the entry has a long length. Suppose we know that Q^9 has length 5. Then Q = 3 since $2^9 = 512$, $3^9 = 19683$ and $4^9 = 262144$.

In a published puzzle, we are told that an entry of length 4 is a square, a cube and a fourth power. This means that it is also a twelfth power and the only one of length 4 is $2^{12} = 4096$.

The situation with factorials is even more informative. Only 4! has length 2; 5! and 6! have length 3, while all larger factorials have different lengths. In a published puzzle, $B! - IG$ has entry length 6 and all symbols represent one of 1–26. The largest possible value for IG is 650, so B must be 9 and the entry is 362□□□, since the next digit of 9! is larger than 6.

3. Symbolic Powers

If the clues contain several instances of, say X^S and X^T, for various X then it is likely that S and T are 2 and 3 in either order. A study of the lengths of the relevant entries may confirm this and point to the correct choice. One-off power symbols may represent a larger number, but then the methods in items 1 and 2 may come into play.

In a published puzzle, P^4 has entry length 4 and P^T has length 3. This means that T is 2 or 3. The puzzle also contains four instances of symbols to the power of W, with entry lengths 3, 2, 3, 3, which strongly suggests W = 2 and hence T = 3, with further information on P to follow.

4. Links

Look out for the same symbol appearing in a simple form in more than one clue, especially if the entries intersect. For example, suppose that 1 Across is E^4, length 4, and that 2 Down (starting in the second square) is E^2, length 2. The across clue tells us that E must be between 6 and 9 inclusive. The down clue tells us that it must be between 4 and 9 inclusive, but together we can deduce that E = 7, since only $7^4 = 2401$ and $7^2 = 49$ provide the correct intersection.

The links need not intersect. Suppose that F^3 has length 4, while F^7 has length 10. The first tells us that F must be at most 21, while the second tells us that F must be at least 20. Hence F is 20 or 21. For the former, the larger answer will contain seven zeros and there is a good chance that we can eliminate it on that account: see item 6.

5. Palindromes

These occur frequently in puzzles, in two ways. Some answers may be shown to be palindromes by a label or by the fact that its clue is also palindromic. In this second case the preamble must state that this is the case. Particularly well constructed puzzles may be able to state that *only* such palindromic clues lead to palindromic answers.

Should palindromes be a feature – no matter how they are indicated – they are a valuable solution aid. Thus, a palindromic perfect square of length 3 must be one of 121, 484 or 676.

One published puzzle contains only palindromes that are divisible by their length. This means that entries of length two must be 22 or 44 or 66 or 88.

6. No Leading Zero
This property, assuming it can be relied upon, should be borne in mind throughout the solution since it can remove some competing entries at any stage. For example, in item 4 we had four possible answers for F^4 of length 4: `1296`, `2401`, `4096` and `6561`. If there are down entries emanating from the second and third digits, we could immediately rule out F = `7` and F = `8`, due to the zeros in their fourth powers.

7. Symbolic Clue Numbers
This device, should it be used, tends to make the puzzle a little harder. But it may yield information in the early stages of the solution. For example, grids usually contain very few places where there is an across and a down entry starting from the same square. Their clue numbers must correspond to symbols for which there is more than one clue.

Some more advanced puzzles replace numbered clues by sets of symbolic expressions that represent a connected set of numbers. (These may use more mathematical concepts such as those in Section 6.) Occasionally these are listed in the order in which the solver should tackle them – or even the reverse order! – but neither indication should be assumed until there is supporting evidence. The most usual starting point involves a set where there is a repeated symbol or a pair of symbols that label intersecting grid entries of short length. In this type of puzzle, reversals of grid entries are common, for which the usual notation is J′ for the reverse of J. A set involving J and J′ could prove just as useful as one involving J twice.

4. Developing the Solution

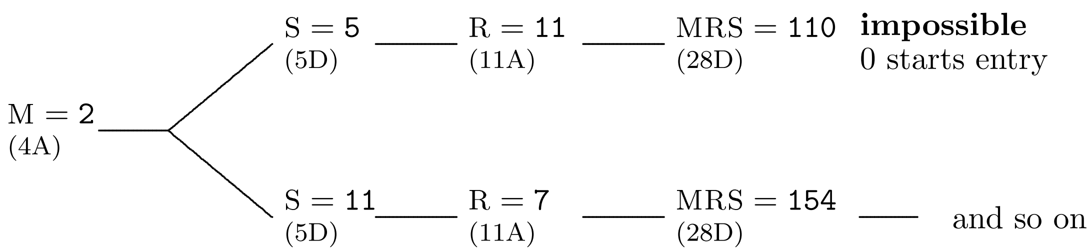
Once a start has been made, the solution tends to develop in two ways. A value will have been identified for one symbol. That can be used in other clues and may shed light on further symbols using the same methods as in the previous section. For example, if a value has been found for E, then a clue SESS effectively becomes a cubic power.

At the same time, the solver may have been able to enter numbers into the grid. The digits where these intersect other entries may be exploitable to provide further information, in ways that will be explored later in this section.

It is possible that the solver will be able to travel along a clear path, identifying more symbols and entering more into the grid, perhaps even to the end of the puzzle. It is more likely, however, that a point will be reached where there is no certain deduction and a set of alternatives appear: the ‘middlegame’.

It is likely that all but one of these will lead to a contradiction, but perhaps after a significant amount of work. There are different methods for dealing with this. Some solvers have a gut feeling for the correct choice and will follow it in the hope that no contradiction appears. If this fails, they will most likely have to rub out their entries and start afresh, since it is difficult to disentangle the correct from the erroneous. (Note that this may happen if an error is made in calculation or transcription, even with correct logic.)

Another technique is to maintain careful records. A 'tree' diagram could be constructed, with a fork wherever a choice is made and a record of what has been calculated along each branch and from which clue. A fragment of such a tree could be the following:



This is time consuming and potentially costly in paper. A middle road is to follow one's instincts, but to annotate the main record of symbol values with a counter showing the order in which they were identified and perhaps the clue involved. The solver may be able to backtrack without having to start again from scratch.

Once numbers start to appear in the grid, a wide range of tools become available. The following items illustrate 10 of these. Each may, on its own, deliver only a small grain of information, but in conjunction with one or more of the others that may be enough to determine the value of a symbol.

As more and more numbers start to appear in the grid, it will become as common to deduce information about clues from partial grid entries as to deduce entries from clues. Several of the items that follow are more useful in such reverse working.

1. Ordering

Since it is almost certain that all answers are positive numbers, any subtraction will yield information. Thus, a clue such as $J(K - L)$ tells us that K is greater in value than L.

2. No Leading Zero

This has been mentioned several times already. If anything, it grows in importance as the grid fills.

3. No Repeated Entries

This can occasionally close off a branch of the tree. It may be helpful in a large puzzle if there are several squares or cubes of small size, since there are a limited number of these.

4. Palindromes

If this is a feature of the puzzle, it should be constantly monitored. These clues may prove easier to crack because of the reduced number of possible entries. One may be able to eliminate a possible contender if the preamble states that only palindromic clues lead to palindromic entries.

Another useful contribution occurs when a number is known to be palindromic and a crossing entry fills in one of its digits. Then, unless it is a central digit, it will be possible to fill in its mirror image. This new digit may also appear in a crossing entry and provoke the entry of its mirror image and so on, until we arrive back where we started or at a central digit or a square with no crossing entry.

This is a vital part of any puzzle where every entry is palindromic; once a digit is entered, it can spread throughout the puzzle. In a situation where there is more than one option, it can quickly rule out some possibilities, e.g., by a 0 propagating to the start of an entry.

5. Symbolic Clue Numbers

These were mentioned in the previous section, but a further use arises as the solution develops. Suppose that we have deduced sufficient symbols to be able to compute the value of clue G and that the entry is 193472, say. Then G must be one of the grid numbers for which the entry has length 6. Further, if a partial entry in such a location is significantly consistent with 193472 we may be able to conjecture the value of G. A more complicated situation in which progress can be made is when G itself is involved in the clue. Then trial and error with available grid numbers may deliver a match.

6. Squares

The first and last digits of answers are particularly important. The first digit dictates the size and we shall take advantage of that in item 10. The final digit is frequently involved in logical arguments, based on facts such as the following. If the answer is known to be a square then it has one of only six possible endings:

0 ($?0^2$), 1 ($?1^2$, $?9^2$), 4 ($?2^2$, $?8^2$), 5 ($?5^2$), 6 ($?4^2$, $?6^2$), 9 ($?3^2$, $?7^2$),

where '?' can be any sequence of digits.

7. Primes

Primes can only end in 1, 3, 7 or 9. This fact, and the table in the *Appendix*, can help in puzzles where primality is a feature.

8. Divisibility

There are several rules for determining whether one number is divisible by another. The simplest of these are set out below:

2: the last digit must be even;

3: the sum of the digits must be divisible by 3,
3 divides 72318: $7+2+3+1+8=21=3\times 7$;

4: the last two digits must be divisible by 4,
4 divides 9156: $56=4\times 14$;

5: the last digit must be 0 or 5;

8: the last three digits must be divisible by 8,
8 divides 82328: $328=8\times 41$;

9: the sum of the digits must be divisible by 9,
9 divides 64035: $6+4+0+3+5=18=9\times 2$.

The following example shows how this last case can be put to use. Suppose that the grid entry for ASSESS is 4□□87 and that we know S = 3. Then 81 × AE must be divisible by 9. This means that the two missing digits in the grid entry add up to 8 or 17, to ensure the overall digit total is 27 or 36. The only possibilities are 08, 17, 26, 35, 44, 89 and their reversals. It takes little time on a calculator to verify that only 42687 is divisible by 81. Further, the quotient is 527, for which the table in the *Appendix* shows the prime factors to be 17 and 31. If the number 1 is ruled out, these must be the values of A and E in some order.

The case of 5 is worthy of note. Once a component ending in 0 or 5 is identified, all its multiples must end in the same. This can move quite rapidly through the grid. In some cases we can deduce an entry of 5 since the alternative would mean a crossing entry starting with zero.

9. Divisors

Suppose we know the final digit of the entry for RS and that R ends in the digit n. We can immediately deduce the last digit of S in the cases n = 1, 3, 7, 9 and have the choice of only two possibilities in the cases n = 2, 4, 6, 8. This deduction is made as if we were dividing some 2-digit number by n.

For example, if RS = 1□7□4 and R = □9, then we note that $54 = 9 \times 6$ and conclude that S = □□6. (The 5 in 54 has no significance because of possible 'carries' in the rest of the calculation.)

Another example is RS = 1□7□4 with R = □8. Then we note that $24 = 8 \times 3$ and $64 = 8 \times 8$, concluding that S = □□3 or □□8. We may have some other information that will dictate which is the correct version.

10. Estimation

This technique is wide-ranging and powerful. It may be required in a particularly difficult puzzle or when one has lost the setter's intended thread. It is also helpful in the final phase of the solution.

When a grid entry is partially known, we can deduce the range of possible values by replacing all unknown digits by 0 or by 9, assuming the puzzle uses base 10. (A starting digit must be replaced by 1 rather than 0.) Applying known information in the clue will, in turn, deliver information about some of the unknown symbols. In some cases this may be sufficient to determine their values. This is best seen by example.

Suppose that we know EDDY = 2□9□□, where E = 7 and Y = 11. Then DD can be bounded between

$$20900/77 = 271.42\cdots \qquad \text{and} \qquad 29999/77 = 389.59\cdots$$

and since it is a square, it must be one of 17^2, 18^2 or 19^2. It is straightforward to check that only 18^2 will reproduce the middle 9, so D = 18 and EDDY = 24948.

In some cases it is quicker to use the digit 5 to replace unknown digits, so that one operates in the middle of the range. For example, suppose CRAB = □831□, where R = 37, A = 7 and B = 11. Now $58315/(37\times7\times11) = 20.46\cdots$. If we further know that all symbols represent prime numbers, there are only a few possibilities to try and C = 31 is soon discovered, giving CRAB = 88319.

This example was potentially difficult since one knew neither an estimate of the size (no first digit) nor any divisibility information (no final digit). The next two examples – with a larger entry length and proportionately less information – show how we can achieve more if we do have that information.

We stay with CRAB, with R, A and B as above, but dispense with the primality restriction. Suppose that CRAB = 4□87□□. In this case it is a good tactic to test each of 4□8755/2849, replacing □ by 0, 1, . . . , 9 to see if a near whole number appears. We find $438755/2849 = 154.0031\cdots$ is the nearest and delivers a feasible result with C = 154. Other nearby whole numbers are 147, 161, 168 and 175, but none matches the 7 in the answer after recomputing CRAB.

Now suppose we have CRAB = □34□□7. Since RAB = 2849, an application of item 9 shows that C must end in 3. Bounding information is

$$134007/2849 = 47.03\cdots \quad \text{and} \quad 934997/2849 = 328.18\cdots$$

so we have 28 possible numbers of the form (□)□3 to check. It is likely to be quicker to test each of □34557/2849 for an answer near a whole number ending in 3. This uncovers 8 as the only possibility, so that C = 293 and CRAB = 834757.

There is one final observation to make about this method. Those who rework these last calculations will discover that the number 2849 must be keyed repeatedly. This can be avoided by making use of the memory facility in one's calculator, to store that number once and bring it back quickly when needed.

As the grid fills up and more symbol values are determined the solver will have more options available and the problem shifts to deciding which will be the most efficient. Indeed, at some stage all symbols may be fixed and the grid can be filled by straightforward computation. (In practice, it is often the case that a few symbols appear only once and will be determined as those final clues are solved.) Whatever the case, the solver will have many of the devices above to help in the 'endgame' phase.

Finally, it is good policy to perform a check once the grid has been completed. Revisit the clues in order, recalculate their values using those determined for the symbols involved and check that their grid entries are correct.

5. Number Bases

The numbers we use in everyday life are expressed 'in base 10', related to the fact that we have single digit representations only for the first 10 numbers, 0–9. But other number bases are possible – computers use base 2 or base 16 – and mathematical crossword setters have made use of this to extend the range of their puzzles. There have been puzzles that use a few different bases and ones that use a different base in each row and a different base in each column. This is usually explained in the preamble, although there are puzzles that use a single base throughout, with no explanation other than some oblique hint in the title or the preamble. This section finishes with some comments on how to tackle puzzles where number bases must be deduced.

If this is a feature, it is essential to know how to translate between different bases. (The *Appendix* contains a table that gives each number in the range 1–1000 in bases from 4 to 12, with instructions on how to use the table for bases 2, 3 and 16.)

We concentrate on how to convert to and from base 10. Conversion between other bases can be done using base 10 as an intermediary. Only whole numbers are treated, although fractions can be expressed in forms other than 'decimal'.

There is a useful piece of notation that helps in setting out examples, although it is not used in the puzzles themselves. When different bases are in use, we can specify the base by listing it as a subscript after the number. Thus, we have $18_{10} = 10010_2 = 200_3 = 12_{16}$.

On the subject of notation, we require single symbols for each number less than the base, just as we use 0–9 for base 10. For bases less than 10 we merely use these familiar decimal digits, but for bases larger than 10 we require single symbols to represent 10, 11, etc. The most common convention – not always adopted by setters who may have to avoid clashes with the rest of the puzzle – is to use the letters `A`, `B`, etc. Thus *hexadecimal* (base 16) uses `A` = 10, . . . , `F` = `15`.

There is one final remark to make about notation. Mathematicians regularly represent sequences using subscripts, e.g., x_0, x_1, x_2, ... This allows them to talk about the number $d_1 d_2 \cdots d_n$ when devising formulae and calculation techniques for 'general' numbers. We shall avoid this, choosing a simpler notation at the cost of not dealing with a general sequence length. It should, however, be clear how to convert the methods to other lengths.

We start with the easier problem, that of converting from base x to base 10. This is done directly from the meaning of the number notation. In that base, the number *fedcba* represents

$$n = fx^5 + ex^4 + dx^3 + cx^2 + bx + a.$$

Here, each of f, e, d, c, b, a must lie between 0 and $x - 1$. To convert to base 10, replace x and each of these digits, if necessary, by their base 10 equivalents and evaluate the given expression. Thus,

$$21063_7 = 2 \times 7^4 + 1 \times 7^3 + 0 \times 7^2 + 6 \times 7 + 3 = 5190_{10}.$$

There is an alternative method, related to the mathematical device of *nested multiplication* or *synthetic division*, which is more efficient. We start with the leading digit, multiply it by the base and add the next digit, then repeat this until the final digit is used. Thus, for the number *fedcba*, we use the following key presses on the calculator:

$$f \boxed{\times} x \boxed{+} e \boxed{=} \boxed{\times} x \boxed{+} d \boxed{=} \boxed{\times} x \boxed{+} c \boxed{=} \boxed{\times} x \boxed{+} b \boxed{=} \boxed{\times} x \boxed{+} a \boxed{=}.$$

Thus, for the hexadecimal number 3A50E, we have

$$\begin{aligned} 3 \times 16 + 10 &= 58, \\ 58 \times 16 + 5 &= 933, \\ 933 \times 16 + 0 &= 14928, \\ 14928 \times 16 + 14 &= 238862, \end{aligned}$$

giving the answer 238862 in base 10.

To convert from base 10 to base x, we effectively reverse this second method, repeatedly dividing by the base (instead of multiplying) and 'subtracting' (recording and ignoring) the remainders, which form the digits in the new base. The key calculation is:

$$n \text{ divided by } x \text{ gives quotient } q \text{ and remainder } r,$$

with r recorded and q replacing n. The calculation is repeated until q is zero. The number in base x is then constructed by juxtaposing the remainders in the reverse order of their calculation. For example, to convert 3413 to base 6, we find

3413/6:	quotient	568	remainder	5
568/6:	quotient	94	remainder	4
94/6:	quotient	15	remainder	4
15/6:	quotient	2	remainder	3
2/6:	quotient	0	remainder	2

Hence $3413_{10} = 23445_6$. Another example is 9371, to be converted to base 11:

9371/11:	quotient	851	remainder	10
851/11:	quotient	77	remainder	4
77/11:	quotient	7	remainder	0
7/11:	quotient	0	remainder	7

Hence $9371_{10} = 704\text{A}_{11}$.

It is usually the case in puzzles with different number bases that the bases must be deduced as part of the solution. The principal tool for this is size. Numbers get shorter in length as the base increases. Conversely, numbers in *binary* (base 2) or *ternary* (base 3) are much longer than in decimal.

A clue with many multiplications and a short entry length is likely to be entered using a large base. A clue that uses only addition or, even more so, subtraction is likely to be entered in a small base.

If the decimal version of an entry is known or guessed, it is helpful to compare it with x^n and x^{n-1} for various x, where n is the entry length. It must lie between these numbers if base x is feasible, assuming that the entry cannot start with zero. For example, consider the decimal $n = 13421$ with an entry length of 7. The smallest base 5 entry is $1\,000\,000_5 = 5^6 = 15625_{10}$, which is larger than n, so base 5 is not feasible. But $4^6 = 4096$ and $4^7 = 16384$, which encompass n, so base 4 is feasible. Finally, $3^7 = 2187$ is smaller than n, so base 3 would require more than 7 digits. This means the base must be 4 and the entry is 3101231.

In a published puzzle, we are told that each symbol is one of the first six prime numbers and that GGG has a palindromic entry, of length 4. This greatly cuts down the possibilities. There are four, but surprisingly the grid entry is the same for each: 1331. (The values for G are 5 (base 4), 7 (base 6), 11 (base 10) and 13 (base 12).)

Partial grid entries are not as helpful as in decimal puzzles since divisibility rules are not straightforward. Indeed it is possible to have **even** decimal numbers that *appear* to be **odd** in an odd number base: $22_{10} = 211_3$.

One useful observation is that the number base must exceed any digit already in place. Thus, if an entry appears as 6□□9□, it must represent a number in a base larger than 9. If the puzzle allows bases up to and including 12, the possible *decimal* ranges for the entry are, for each base:

60090–69999 (base 10), 87945–102475 (base 11), 124524–145127 (base 12).

These (non-overlapping) ranges may help determine which base is to be used. On the other hand, a row that contains to date only 0 and 1 *may* turn out to contain binary numbers. There is, however, a warning to sound here. This device does not carry through to isolated squares that do not belong to an entry in a row or column. Thus a binary row may contain a square holding 6 if it belongs only to an intersecting column entry.

6. Advanced Topics

The following topics lie in the field of mathematics rather than arithmetic. They have been selected since they have already appeared in puzzles and are fruitful enough to make reappearance a possibility.

Algebraic calculations

Some puzzles demand a higher level of algebra than that employed in the preceding sections. This was certainly true of the series of 45 puzzles composed by *Rhombus* for the *Listener* between 1960 and 1980. Two examples of the underlying mathematics are given here.

The P and S Game This involves a quantity called the P/S value of the numbers P and S, defined by P + S + PS. One method of making progress is to note that

$$p + s + ps = 1 + p + s + ps - 1 = (1 + p)(1 + s) - 1,$$

so that adding one to the P/S number gives a number with useful factors.

Can You Do Division? This is one of the most elegant mathematical puzzles ever published. The clue for each grid entry is merely a count of the number of its divisors. The method for making progress is to use prime number factorisation, since

$$n = p^a q^b r^c s^d \cdots \quad \text{has a count of divisors:} \quad (a+1)(b+1)(c+1)(d+1)\cdots.$$

Greatest Common Divisor & Least Common Multiple

For a pair of (whole) numbers m and n, there are two numbers that are important in mathematics and play a rôle in some puzzles: $\gcd(m,n)$: their *greatest common divisor*, and $\operatorname{lcm}(m,n)$: their *least common multiple*. (The former is sometimes called the *highest common factor* and written $\operatorname{hcf}(m,n)$; upper case letters may also be used.)

$\gcd(m,n)$ is the largest number that divides both m and n.

$\operatorname{lcm}(m,n)$ is the smallest number that is divisible by both m and n.

The lcm is used for adding fractions, while the gcd is useful for computing the lcm.

Indeed, there is a sequence of calculations, called the *Euclidean Algorithm*, which will find the gcd for any pair of numbers. Using the gcd, we have

$$\operatorname{lcm}(m,n) = m \times n / \gcd(m,n).$$

For mathematical puzzles, however, the following method is more useful. It is based on the prime factorisation of m and n and constructs the factors of $\gcd(m,n)$ and $\operatorname{lcm}(m,n)$ from them.

The prime factors of $\gcd(m,n)$ are those that are common to m and n; the power of each factor is the *lesser* of the powers of the corresponding prime in m and n.

The prime factors of $\operatorname{lcm}(m,n)$ are those that are in either m or n or both; the power of each is the *greater* of the powers of the corresponding prime in m and n (counting an absent prime as having power zero).

For example, consider

$$m = 24 = 2^3 \cdot 3, \qquad n = 84 = 2^2 \cdot 3 \cdot 7.$$

Then

$$\gcd(m,n) = 2^2 \cdot 3 = 12, \qquad \operatorname{lcm}(m,n) = 2^3 \cdot 3 \cdot 7 = 168.$$

Finally, the numbers m and n are called *relatively prime* if $\gcd(m,n) = 1$. This means that they have no factor in common (other than 1). This condition is often imposed in mathematics to remove solutions that are scaled versions of ones already found. It can also occur in puzzles and, if stated in the preamble, it is normally helpful to the solver.

Sequences

There are many interesting sequences of numbers in mathematics, some of which have appeared in mathematical puzzles.

One of the most famous, which has appeared in word-based puzzles as well as numerical ones, is the *Fibonacci sequence*. The first two terms are both 1 and thereafter each term is the sum of the previous two. Hence the sequence proceeds $1+1=2$, $1+2=3$, $2+3=5$, and so on:

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233,\ \ldots.$$

This sequence is held to model various natural phenomena.

An *arithmetic sequence* or *arithmetic progression* is one where consecutive terms differ by the same amount, such as 2, 4, 6, 8, ... or 15, 12, 9, 6, ... Some puzzles require the sum of such sequences, for which there is a simple rule:

$$\text{sum of } n \text{ terms} = \text{average of first and last terms} \times \text{ number of terms.}$$

Thus the sum of the odd numbers less than 100 is $\frac{1}{2}(1+99) \times 50 = 2500$.

A common arithmetic sequence is the sum of the first n whole numbers, for which this rule gives

$$T_n = 1 + 2 + 3 + \cdots + (n-1) + n = \tfrac{1}{2}n(n+1).$$

The notation T_n has been chosen since this is the n^{th} *triangular* number. These count the number of dots in arrangements such as . The sequence of T_n is:

$$1,\ 3,\ 6,\ 10,\ 15,\ 21,\ 28,\ 36,\ 45,\ 55,\ 66,\ 78,\ \ldots.$$

Similarly, we could seek *square* and *cube* numbers as the count of dots required to construct those shapes. These are just the familiar squares and cubes. We can add these up, just as we did for $1 + \cdots + n$:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \tfrac{1}{6}n(n+1)(2n+1),$$
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \tfrac{1}{4}n^2(n+1)^2.$$

(Note the intriguing outcome that the sum of the first n cubes is the square of the sum of the first n numbers.)

Pythagorean Triples

The triple of whole numbers (a, b, c) is *Pythagorean* if they represent the sides of a right-angled triangle. This translates from geometry to algebra as $a^2 + b^2 = c^2$. There is a guaranteed method for finding all triples of this kind:

$$a = m^2 - n^2, \qquad b = 2mn, \qquad c = m^2 + n^2.$$

Any choice of m and n, with m larger than n, will give a valid triple. If, however, we ensure that one is even and one is odd and they have no common factor, then a, b and c are also guaranteed to have no common factor. All other triples will be multiples of one of these.

The following are the triples with the smallest numbers:

$m = 2$,	$n = 1$	:	$a = 3$,	$b = 4$,	$c = 5$
$m = 3$,	$n = 2$	:	$a = 5$,	$b = 12$,	$c = 13$
$m = 4$,	$n = 1$	:	$a = 15$,	$b = 8$,	$c = 17$
$m = 4$,	$n = 3$	:	$a = 7$,	$b = 24$,	$c = 25$

Quadratic Equations

Equations – expressions containing an '=' and an unknown quantity – are powerful tools, often enabling possible values of that quantity to be found. They range from the straightforward, such as those used in Section 4 to complete partially solved clues, to the intractable. Occupying the middle ground are *quadratic* equations, where the square of the unknown quantity is involved. These are within the scope of possible *Listener* puzzles but are also a useful tool in the solution process as an example will show.

The most general quadratic equation is $ax^2 + bx + c = 0$, where a, b and c are known values and x is to be determined. There are two solutions:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \text{and} \qquad \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

In *some* cases the second of these is negative, in which case it is unlikely to be relevant in one of these puzzles.

Note that we require $b^2 - 4ac$ to be zero or greater than zero if there is to be a solution, a fact that could be helpful. Also, assuming that any solution will need to be a whole number, $b^2 - 4ac$ would have to be a perfect square, which is also potentially helpful. (This important quantity is known as the *discriminant.*)

Suppose we wish to find L from $\text{PLAY} + \text{BALL} = \mathtt{12331}$, with $\text{P} = \mathtt{7}$, $\text{A} = \mathtt{11}$, $\text{Y} = \mathtt{3}$ and $\text{B} = \mathtt{2}$. The equation that results is $231\text{L} + 22\text{L}^2 = 12331$. Before using the formula above, note that A occurs in both terms, so we would expect a common factor of 11 in the equation. (This may have been used already to fill gaps in 12331.) This can be cancelled to produce the simpler version

$$2\text{L}^2 + 21\text{L} - 1121 = 0.$$

The formula leads to one positive and one negative solution. The former is

$$\text{L} = \frac{1}{2 \times 2}\left(-21 + \sqrt{9409}\right) = \frac{1}{4}\left(-21 + 97\right) = \mathtt{19}.$$

Logarithms

These are very important tools in mathematics, enabling scientists to transform many natural processes to more tractable forms. They occasionally occur in mathematical crosswords.

A *log(arithm)* requires a *base* to be specified. For our purposes the most appropriate base is that of the number system, so here we use base 10. (It should prove straightforward to translate the following to other bases.) The definition is

$$x = \log_{10} n \qquad \text{if} \qquad 10^x = n.$$

(We now drop the subscript 10.)

Some examples are

$\log 100 = 2$	since	$10^2 = 100$
$\log 10000 = 4$	since	$10^4 = 10000$
$\log 10 = 1$	since	$10^1 = 10$
$\log 1 = 0$	since	$10^0 = 1$
$\log 0.1 = -1$	since	$10^{-1} = 0.1$

Logarithms do not play a major rôle in crosswords since there are few useful values that are whole numbers.

Scientific calculators can be used to evaluate logs, but care is required since they normally have two keys, typically marked `log` and `ln`. It is the former that delivers base 10 values. (The other provides the *natural* or *Napierian* logs, which have neater mathematical properties but a base that is not a whole number.)

Scientific calculators also have a key $\boxed{10^x}$ that will 'undo' this. (Sometimes it is resident on the same key as `log` and is accessed using the `shift` key.) If $\log n = x$, applying this key to x will recover n.

Special Numbers

There are three special numbers that have appeared in puzzles. None is a whole number, indeed all three have never-ending and never-repeating decimal expansions. They are, to 50 decimal places,

$\pi =$ 3.14159 26535 89793 23846 26433 83279 50288 41971 69399 37510 $\cdots$

$e =$ 2.71828 18284 59045 23536 02874 71352 66249 77572 47093 69995 $\cdots$

$\phi =$ 1.61803 39887 49894 84820 45868 34365 63811 77203 09179 80576 $\cdots$

π is the ratio of the circumference of a circle to its diameter.

e is the base of the natural logarithms (mentioned in the previous item).

ϕ, equal to $\frac{1}{2}(1+\sqrt{5})$, is the *golden ratio* or *golden mean*, held to measure the ratio of the sides of the most aesthetically pleasing rectangular shape. Ratios of successive terms in the Fibonacci sequence approach ever closer to ϕ.

7. References

Many of these topics are contained in books on 'recreational mathematics', written with the non-specialist in mind. The following two are particularly useful for solving mathematical puzzles.

Wells, Davis, *The Penguin Dictionary of Curious and Interesting Numbers*, Penguin

This has occasionally been recommended in the series of *Listener* puzzles.

Jenkins, Adrian, *The Number File*, Tarquin Press

This is used by one of the most prolific current setters.

The following two take some of the ideas further and may be enjoyed by those who have been intrigued by the topics in these 'Notes'.

Conway, John H. & Guy, Richard K., *The Book of Numbers*, Springer-Verlag

Beiler, Albert, *Recreations in the Theory of Numbers*, Dover